MATEMÁTICA A - 12º Ano
Probabilidades - Demonstrações
Propostas de resolução

Exercícios de exames e testes intermédios

1. Temos que,

$$\begin{aligned}
P\left(A\cup\overline{B}\right)-1+P(B) &= P(A)+P\left(\overline{B}\right)-P\left(A\cap\overline{B}\right)-1+P(B) && (1)\\
&= P(A)-P\left(A\cap\overline{B}\right)-1+P(B)+P\left(\overline{B}\right)\\
&= P(A)-P\left(A\cap\overline{B}\right)-1+1 && (2)\\
&= P(A)-P\left(A\cap\overline{B}\right)\\
&= P(A)-\Big(P(A)-P(A\cap B)\Big) && (3)\\
&= P(A)-P(A)+P(A\cap B)\\
&= P(A\cap B)\\
&= P(A\cap B)\times\frac{P(A)}{P(A)} && (4)\\
&= P(A)\times\frac{P(A\cap B)}{P(A)}\\
&= P(A)\times P(B|A) && (5)
\end{aligned}$$

(1) Teorema: $P\left(X\cup Y\right)=P(X)+P(Y)-P(X\cap Y)$
(2) Teorema: $P(X)+P\left(\overline{X}\right)=1$
(3) Teorema: $P\left(X\cap\overline{Y}\right)=P(X)-P(X\cap Y)$
(4) Hipótese: $P(A)\neq 0$
(5) Definição: $P(X|Y)=\dfrac{P(X\cap Y)}{P(Y)}$

Logo, $P\left(A\cup\overline{B}\right)-1+P(B)=P(A)\times P(B|A)$ *q.e.d.*

Exame – 2015, 2ª Fase

2. Como A e $\overline{A}$ são acontecimentos equiprováveis, temos que $P(A) = P\left(\overline{A}\right)$
Como $P(A) = 1 - P\left(\overline{A}\right)$ vem que

$$P(A) = 1 - P(A) \Leftrightarrow P(A) + P(A) = 1 \Leftrightarrow 2P(A) = 1 \Leftrightarrow P(A) = \frac{1}{2}$$

Como A e B são acontecimentos independentes, temos que

$$P(A \cap B) = P(A) \times P(B)$$

Assim,

$2P(A \cup B) = 2\Big(P(A) + P(B) - P(A \cap B)\Big)$	Teorema: $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$
$= 2P(A) + 2P(B) - 2P(A \cap B)$	
$= 2P(A) + 2P(B) - 2 \times P(A) \times P(B)$	Hipótese: $P(A \cap B) = P(A) \times P(B)$
$= 2 \times \frac{1}{2} + 2P(B) - 2 \times \frac{1}{2} \times P(B)$	Hipótese: $P(A) = \frac{1}{2}$
$= 1 + 2P(B) - 1 \times P(B)$	
$= 1 + P(B)$	

Logo, se $P(A) = P\left(\overline{A}\right)$ e $P(A \cap B) = P(A) \times P(B)$, então $2P(A \cup B) = 1 + P(B)$ *q.e.d.*

Exame – 2014, Ép. especial

3. Como A e B são acontecimentos incompatíveis, temos que $P(A \cap B) = 0$ e que $P(A \cup B) = P(A) + P(B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)} \underset{\text{Hip.}}{=} \frac{0}{P(B)} = 0$

Como, $P(A) \neq 0$, então $P(A|B) < P(A)$

Pelo teorema $P\left(A \cap \overline{B}\right) = P(A) - P(A \cap B)$, e como $P(A \cap B) = 0$, vem que:
$P\left(\overline{B}|A\right) = \frac{P\left(\overline{B} \cap A\right)}{P(A)} = \frac{P(A) - P(A \cap B)}{P(A)} = \frac{P(A) - 0}{P(A)} = 1$

Como $P(A \cup B) = P(A) + P(B)$ e $P(B) \neq 0$, então $P(A) \neq 1$, pois $P(A \cup B) \leq 1$, pelo que $P(A) < P\left(\overline{B}|A\right)$

Logo $P(A|B) < P(A) < P\left(\overline{B}|A\right)$ *q.e.d.*

Teste Intermédio 12º ano – 29.11.2013

4. Como A e B são acontecimentos independentes, temos que

$$P(A \cap B) = P(A) \times P(B)$$

Assim,

$$\begin{aligned}
P\left(\overline{A} \cap B\right) + P\left(\overline{A}\right) \times \Big(1 - P(B)\Big) &= P\left(\overline{A} \cap B\right) + \Big(1 - P(A)\Big) \times \Big(1 - P(B)\Big) && (1)\\
&= P\left(\overline{A} \cap B\right) + 1 - P(B) - P(A) + P(A) \times P(B) \\
&= P\left(\overline{A} \cap B\right) + 1 - P(B) - P(A) + P(A \cap B) && (2)\\
&= P\left(\overline{A} \cap B\right) + P(A \cap B) + 1 - P(B) - P(A) \\
&= P(B) + 1 - P(B) - P(A) && (3)\\
&= 1 - P(A) \\
&= P\left(\overline{A}\right) && (1)
\end{aligned}$$

(1) Teorema: $P\left(\overline{X}\right) = 1 - P(X)$
(2) Hipótese: $P(A \cap B) = P(A) \times P(B)$
(3) Teorema: $P(X) = P(X \cap Y) + P\left(X \cap \overline{Y}\right)$

Logo $P\left(\overline{A} \cap B\right) + P\left(\overline{A}\right) \times \Big(1 - P(B)\Big) = P\left(\overline{A}\right)$ *q.e.d.*

Exame – 2012, Ép. especial

5. Temos que,

$$\begin{aligned}
P\left(\overline{A \cap B}|B\right) + P(A|B) &= \frac{P\left(\left(\overline{A \cap B}\right) \cap B\right)}{P(B)} + \frac{P(A \cap B)}{P(B)} && \text{Definição: } P(X|Y) = \frac{P(X \cap Y)}{P(Y)}\\
&= \frac{P\left(\left(\overline{A} \cup \overline{B}\right) \cap B\right) + P(A \cap B)}{P(B)} && \text{Leis de De Morgan: } \overline{X \cap Y} = \overline{X} \cup \overline{Y}\\
&= \frac{P\left(\left(\overline{A} \cap B\right) \cup \left(\overline{B} \cap B\right)\right) + P(A \cap B)}{P(B)} \\
&= \frac{P\left(\overline{A} \cap B\right) + P(A \cap B)}{P(B)} && \overline{B} \cap B = \emptyset \text{ e } X \cup \emptyset = X\\
&= \frac{P(B) - P(A \cap B) + P(A \cap B)}{P(B)} && \text{Teorema: } P\left(X \cap \overline{Y}\right) = P(X) - P(X \cap Y)\\
&= \frac{P(B)}{P(B)} \\
&= 1 && \text{Hipótese: } P(B) \neq 0
\end{aligned}$$

Logo, se $P(B) \neq 0$ então $P\left(\overline{A \cap B}|B\right) + P(A|B) = 1$ *q.e.d.*

Exame – 2012, 2ª Fase

6. Temos que,

$$\begin{aligned}
P\left(\overline{A\cap B}|B\right) &= \frac{P\left(\left(\overline{A\cap B}\right)\cap B\right)}{P(B)} && \text{Definição: } P(X|Y)=\frac{P(X\cap Y)}{P(Y)}\\
&= \frac{P\left(\left(\overline{A}\cup\overline{B}\right)\cap B\right)}{P(B)} && \text{Leis de De Morgan: } \overline{X\cap Y}=\overline{X}\cup\overline{Y}\\
&= \frac{P\left(\left(\overline{A}\cap B\right)\cup\left(\overline{B}\cap B\right)\right)}{P(B)} && \\
&= \frac{P\left(\overline{A}\cap B\right)}{P(B)} && \overline{B}\cap B=\emptyset \text{ e } X\cup\emptyset=X\\
&= P(\overline{A}|B) && \text{Definição: } P(X|Y)=\frac{P(X\cap Y)}{P(Y)}
\end{aligned}$$

Logo, se $P(B)\neq 0$ então $P\left(\overline{A\cap B}|B\right)=P(\overline{A}|B)$ *q.e.d.*

Exame – 2011, Ép. especial

7. Temos que,

$$\begin{aligned}
1-\frac{1-P(B)}{P(A)} &= \frac{P(A)}{P(A)}-\frac{1-P(B)}{P(A)} && \\
&= \frac{P(A)-1+P(B)}{P(A)} && \\
&= \frac{P(A)+P(B)-\left(P(A\cup B)+P\left(\overline{A\cup B}\right)\right)}{P(A)} && \text{Teorema: } P(X)+P\left(\overline{X}\right)=1\\
&= \frac{P(A)+P(B)-P(A\cup B)-P\left(\overline{A\cup B}\right)}{P(A)} && \\
&= \frac{P(A\cap B)-P\left(\overline{A\cup B}\right)}{P(A)} && \text{Teorema: } P(X\cap Y)=P(X)-P(Y)-P(X\cup Y)\\
&= \frac{P(A\cap B)}{P(A)}-\frac{P\left(\overline{A\cup B}\right)}{P(A)} && \\
&= P(B|A)-\frac{P\left(\overline{A\cup B}\right)}{P(A)} && \text{Definição: } P(X|Y)=\frac{P(X\cap Y)}{P(Y)}
\end{aligned}$$

Logo $1-\dfrac{1-P(B)}{P(A)}=P(B|A)-\dfrac{P\left(\overline{A\cup B}\right)}{P(A)} \Leftrightarrow P(B|A)=1-\dfrac{1-P(B)}{P(A)}+\dfrac{P\left(\overline{A\cup B}\right)}{P(A)}$

Como $\dfrac{P\left(\overline{A\cup B}\right)}{P(A)}\geq 0$, então $P(B|A)\geq 1-\dfrac{1-P(B)}{P(A)}$ *q.e.d.*

Exame – 2011, 1ª Fase

8. Temos que,

$$\begin{aligned}
P(A\cup B) < P(A|B)\times P\left(\overline{B}\right) &\Leftrightarrow P(A)+P(B)-P(A\cap B) < P(A|B)\times P\left(\overline{B}\right) && (1)\\
&\Leftrightarrow P(A)+P(B) < P(A|B)\times P\left(\overline{B}\right) + P(A\cap B)\\
&\Leftrightarrow P(A)+P(B) < P(A|B)\times \Big(1-P(B)\Big) + P(A\cap B) && (2)\\
&\Leftrightarrow P(A)+P(B) < P(A|B) - P(A|B)\times P(B) + P(A\cap B)\\
&\Leftrightarrow P(A)+P(B) < P(A|B) - \frac{P(A\cap B)}{P(B)}\times P(B) + P(A\cap B) && (3)\\
&\Leftrightarrow P(A)+P(B) < P(A|B) - P(A\cap B) + P(A\cap B)\\
&\Leftrightarrow P(A)+P(B) < P(A|B)
\end{aligned}$$

(1) Teorema: $P(X\cup Y) = P(X)+P(Y)-P(X\cap Y)$

(2) Teorema: $P\left(\overline{X}\right) = 1 - P(X)$

(3) Definição: $P(X|Y) = \dfrac{P(X\cap Y)}{P(Y)}$

Logo, $P(A\cup B) < P(A|B)\times P\left(\overline{B}\right) \Leftrightarrow P(A)+P(B) < P(A|B)$ *q.e.d.*

Teste Intermédio 12º ano – 26.05.2011

9. Temos que,

$$\begin{aligned}
\frac{P(A\cup B)}{P(B)} - P\left(\overline{A}|B\right) &= \frac{P(A\cup B)}{P(B)} - \frac{P\left(\overline{A}\cap B\right)}{P(B)} && \text{Definição: } P(X|Y)=\frac{P(X\cap Y)}{P(Y)}\\
&= \frac{P(A\cup B) - P\left(\overline{A}\cap B\right)}{P(B)}\\
&= \frac{P(A\cup B) - \big(P(B) - P(A\cap B)\big)}{P(B)} && \text{Teorema: } P\left(X\cap\overline{Y}\right) = P(X) - P(X\cap Y)\\
&= \frac{P(A\cup B) - P(B) + P(A\cap B)}{P(B)}\\
&= \frac{P(A)}{P(B)} && \text{Teorema: } P(X) = P(X\cup Y) + P(X\cap Y) - P(Y)
\end{aligned}$$

Logo, se $P(B)\neq 0$ então $\dfrac{P(A\cup B)}{P(B)} - P\left(\overline{A}|B\right) = \dfrac{P(A)}{P(B)}$ *q.e.d.*

Exame – 2010, 2ª Fase

10. Temos que,

$$\begin{aligned}
P(X)\times P(Y|X)+P\left(\overline{X}\right)-P(Y) &= P(X)\times\frac{P(Y\cap X)}{P(X)}+P\left(\overline{X}\right)-P(Y) && (1)\\
&= P(Y\cap X)+P\left(\overline{X}\right)-\left(1-P\left(\overline{Y}\right)\right) && (2)\\
&= 1-P\left(\overline{Y\cap X}\right)+P\left(\overline{X}\right)-1+P\left(\overline{Y}\right) && (2)\\
&= 1-1-P\left(\overline{Y}\cup\overline{X}\right)+P\left(\overline{X}\right)+P\left(\overline{Y}\right) && (3)\\
&= P\left(\overline{X}\right)+P\left(\overline{Y}\right)-P\left(\overline{Y}\cup\overline{X}\right) && \\
&= P\left(\overline{X}\cap\overline{Y}\right) && (4)
\end{aligned}$$

(1) Definição: $P(A|B)=\frac{P(A\cap B)}{P(B)}$
(2) Teorema: $P(A)=1-P\left(\overline{A}\right)$
(3) Leis de De Morgan: $\overline{A\cap B}=\overline{A}\cup\overline{B}$
(4) Teorema: $P(A\cap B)=P(A)+P(B)-P(A\cup B)$

Logo, $P\left(\overline{X}\cap\overline{Y}\right)=P(X)\times P(Y|X)+P\left(\overline{X}\right)-P(Y)$ *q.e.d.*

Teste Intermédio 12º ano – 19.05.2010

11. Temos que,

$$\begin{aligned}
P(A)\times[P(B|A)-1]+P\left(\overline{A}\cup\overline{B}\right) &= P(A)\times\left[\frac{P(A\cap B)}{P(A)}-1\right]+P\left(\overline{A}\cup\overline{B}\right) && (1)\\
&= P(A)\times\frac{P(A\cap B)}{P(A)}-P(A)+P\left(\overline{A\cap B}\right) && (2)\\
&= P(A\cap B)-P(A)+1-P(A\cap B) && (3)\\
&= P(A\cap B)-P(A\cap B)+1-P(A) && \\
&= 1-P(A) && \\
&= P\left(\overline{A}\right) && (3)
\end{aligned}$$

(1) Definição: $P(X|Y)=\frac{P(X\cap Y)}{P(Y)}$
(2) Leis de De Morgan: $\overline{X}\cup\overline{Y}=\overline{X\cap Y}$
(3) Teorema: $P(X)=1-P\left(\overline{X}\right)$

Logo, $P(A)\times[P(B|A)-1]+P\left(\overline{A}\cup\overline{B}\right)=P\left(\overline{A}\right)$ *q.e.d.*

Teste Intermédio 12º ano – 04.12.2009

12. Temos que,

$$
\begin{aligned}
P(B)+P\left(\overline{A}\right)+P\left(\overline{A}\cup\overline{B}\right) &= P(B)+P\left(\overline{A}\right)+P\left(\overline{A}\right)+P\left(\overline{B}\right)-P\left(\overline{A}\cap\overline{B}\right) && (1)\\
&= 2P\left(\overline{A}\right)+P(B)+P\left(\overline{B}\right)-P\left(\overline{A\cup B}\right) && (2)\\
&= 2P\left(\overline{A}\right)+1-P\left(\overline{A\cup B}\right) && (3)\\
&= 2P\left(\overline{A}\right)+1-\Big(1-P(A\cup B)\Big) && (4)\\
&= 2P\left(\overline{A}\right)+1-1+P(A\cup B)\\
&= 2P\left(\overline{A}\right)+P(A\cup B)
\end{aligned}
$$

(1) Teorema: $P(X\cup Y)=P(X)+P(Y)-P(X\cap Y)$
(2) Leis de De Morgan: $\overline{X}\cap\overline{Y}=\overline{X\cup Y}$
(3) Teorema: $P(X)+P\left(\overline{X}\right)=1$
(4) Teorema: $P\left(\overline{X}\right)=1-P(X)$

Logo, $P(B)+P\left(\overline{A}\right)+P\left(\overline{A}\cup\overline{B}\right)=2P\left(\overline{A}\right)+P(A\cup B)$ *q.e.d.*

Exame – 2009, Ép. especial

13. Temos que,

$$
\begin{aligned}
1-P(A|B)\times P(B)-P\left(A\cap\overline{B}\right) &= 1-\frac{P(A\cap B)}{P(B)}\times P(B)-P\left(A\cap\overline{B}\right) && (1)\\
&= 1-P(A\cap B)-P\left(A\cap\overline{B}\right)\\
&= 1-P(A\cap B)-\Big(P(A)-P(A\cap B)\Big) && (2)\\
&= 1-P(A\cap B)-P(A)+P(A\cap B)\\
&= 1-P(A) && (3)\\
&= P\left(\overline{A}\right)
\end{aligned}
$$

(1) Definição: $P(X|Y)=\dfrac{P(X\cap Y)}{P(Y)}$
(2) Teorema: $P\left(X\cap\overline{Y}\right)=P(X)-P(X\cap Y)$
(3) Teorema: $P\left(\overline{X}\right)=1-P(X)$

Logo, $1-P(A|B)\times P(B)-P\left(A\cap\overline{B}\right)=P\left(\overline{A}\right)$ *q.e.d.*

Exame – 2009, 2ª Fase

14. Temos que,

$$
\begin{aligned}
P(A|B)-P(\overline{B})\times P(A|B) &= P(A|B)\times\Big(1-P(\overline{B})\Big)\\
&= P(A|B)\times\Big(1-\big(1-P(B)\big)\Big) && \text{Teorema: } P(X)=1-P\left(\overline{X}\right)\\
&= P(A|B)\times\Big(1-1+P(B)\Big)\\
&= \frac{P(A\cap B)}{P(B)}\times P(B) && \text{Definição: } P(X|Y)=\frac{P(X\cap Y)}{P(Y)}\\
&= P(A\cap B)\\
&= \frac{P(A)\times P(A\cap B)}{P(A)}\\
&= P(A)\times P(B|A) && \text{Definição: } P(X|Y)=\frac{P(X\cap Y)}{P(Y)}
\end{aligned}
$$

Logo, $P(A|B)-P(\overline{B})\times P(A|B)=P(A)\times P(B|A)$ *q.e.d.*

Teste Intermédio 12º ano – 10.12.2008

15. Temos que,

$$\begin{aligned}
1-P\left(\overline{A\cup B}\right)+P(A|B)\times P(B) &= P(A\cup B)+P(A|B)\times P(B) && (1)\\
&= P(A\cup B)+\frac{P(A\cap B)}{P(B)}\times P(B) && (2)\\
&= P(A\cup B)+P(A\cap B)\\
&= P(A)+P(B) && (3)
\end{aligned}$$

(1) Teorema: $P(X)=1-P\left(\overline{X}\right)$

(2) Definição: $P(X|Y)=\dfrac{P(X\cap Y)}{P(Y)}$

(3) Teorema: $P(X\cup Y)+P(X\cap Y)=P(X)+P(Y)$

Logo, $1-P\left(\overline{A\cup B}\right)+P(A|B)\times P(B)=P(A)+P(B)$ *q.e.d.*

Exame – 2008, Ép. especial

16. Temos que,

$$\begin{aligned}
P\left(\overline{A}\right)-P(B)+P(A\cup B) &= 1-P(A)-P(B)+P(A\cup B) && \text{Teorema: } P(X)=1-P\left(\overline{X}\right)\\
&= 1-\Big(P(A)+P(B)-P(A\cup B)\Big)\\
&= 1-P(A\cap B) && \text{Teorema: } P(X\cap Y)=P(X)+P(Y)-P(X\cup Y)\\
&= P\left(\overline{A\cap B}\right) && \text{Teorema: } P(X)=1-P\left(\overline{X}\right)\\
&= P\left(\overline{A}\cup\overline{B}\right) && \text{Leis de De Morgan: } \overline{X\cap Y}=\overline{X}\cup\overline{Y}
\end{aligned}$$

Logo, $P\left(\overline{A}\cup\overline{B}\right)=P\left(\overline{A}\right)-P(B)+P(A\cup B)$ *q.e.d.*

Exame – 2008, 2ª Fase

17. Temos que,

$$\begin{aligned}
P\left(\left(\overline{\overline{A}\cap B}\right)|B\right) &= P\left(\left(\overline{\overline{A}}\cup\overline{B}\right)|B\right) && \text{Leis de De Morgan: } \overline{X\cap Y}=\overline{X}\cup\overline{Y}\\
&= P\left(\left(A\cup\overline{B}\right)|B\right) && \overline{\overline{X}}=X\\
&= \frac{P\Big(\left(A\cup\overline{B}\right)\cap B\Big)}{P(B)} && \text{Definição: } P(X|Y)=\frac{P(X\cap Y)}{P(Y)}\\
&= \frac{P\Big((A\cap B)\cup\left(\overline{B}\cap B\right)\Big)}{P(B)}\\
&= \frac{P(A\cap B)}{P(B)} && \overline{X}\cap X=\emptyset \text{ e } X\cup\emptyset=X\\
&= P(A|B) && \text{Definição: } P(X|Y)=\frac{P(X\cap Y)}{P(Y)}
\end{aligned}$$

Logo, $P\left(\left(\overline{\overline{A}\cap B}\right)|B\right)=P(A|B)$ *q.e.d.*

Teste Intermédio 12º ano – 17.01.2008

18. Como X e Y são acontecimentos independentes, temos que $P(X \cap Y) = P(X) \times P(Y)$
E a probabilidade de que X não ocorra nem ocorra Y pode ser escrita como $P\left(\overline{X} \cap \overline{Y}\right)$
Assim,

$P\left(\overline{X} \cap \overline{Y}\right) = P\left(\overline{X \cup Y}\right)$	Leis de De Morgan: $\overline{A} \cap \overline{B} = \overline{A \cup B}$
$= 1 - P(X \cup Y)$	Teorema: $P\left(\overline{A}\right) = 1 - P(A)$
$= 1 - \Big(P(X) + P(Y) - P(X \cap Y)\Big)$	Teorema: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 1 - P(X) - P(Y) + P(X \cap Y)$	
$= 1 - P(X) - P(Y) + P(X) \times P(Y)$	Hipótese: $P(X \cap Y) = P(X) \times P(Y)$
$= 1 - a - b + a \times b$	Hipótese: $P(X) = a$ e $P(Y) = b$

Logo, $P\left(\overline{X} \cap \overline{Y}\right) = 1 - a - b + a \times b \quad q.e.d.$

Exame – 2007, 2ª Fase

19. Temos que,

$\dfrac{P\left(\overline{B}\right) - P\left(\overline{A} \cap \overline{B}\right)}{P(A)} = \dfrac{P\left(\overline{B}\right) - P\left(\overline{A \cup B}\right)}{P(A)}$	Leis de De Morgan: $\overline{X} \cap \overline{Y} = \overline{X \cup Y}$
$= \dfrac{1 - P(B) - \Big(1 - P(A \cup B)\Big)}{P(A)}$	Teorema: $P\left(\overline{X}\right) = 1 - P(X)$
$= \dfrac{1 - P(B) - 1 + P(A \cup B)}{P(A)}$	
$= \dfrac{-P(B) + P(A \cup B)}{P(A)}$	
$= \dfrac{-P(B) + P(A) + P(B) - P(A \cap B)}{P(A)}$	Teorema: $P(X \cap Y) = P(X) + P(Y) - P(X \cup Y)$
$= \dfrac{P(A) - P(A \cap B)}{P(A)}$	
$= \dfrac{P(A)}{P(A)} - \dfrac{P(A \cap B)}{P(A)}$	
$= 1 - P(B\|A)$	Definição: $P(X\|Y) = \dfrac{P(X \cap Y)}{P(Y)}$

Logo, $\dfrac{P\left(\overline{B}\right) - P\left(\overline{A} \cap \overline{B}\right)}{P(A)} = 1 - P(B|A) \; q.e.d.$

Teste Intermédio 12º ano – 07.12.2005

20. Como A e B são acontecimentos independentes, temos que $P(A\cap B)=P(A)\times P(B)$
Logo pretendemos provar que

$$P(A\cap B)=P(A)\times P(B)\ \Leftrightarrow\ P(B|A)=P\left(B|\overline{A}\right)$$

$P\left(\overline{X}\cap\overline{Y}\right)$
Assim,

$$\begin{aligned}
P(B|A)=P\left(B|\overline{A}\right) &\Leftrightarrow \frac{P(A\cap B)}{P(A)}=\frac{P\left(\overline{A}\cap B\right)}{P\left(\overline{A}\right)} && (1)\\
&\Leftrightarrow P(A\cap B)\times P\left(\overline{A}\right)=P\left(\overline{A}\cap B\right)\times P(A) && (2)\\
&\Leftrightarrow P(A\cap B)\times P\left(\overline{A}\right)=\Big(P(B)-P(A\cap B)\Big)\times P(A) && (3)\\
&\Leftrightarrow P(A\cap B)\times P\left(\overline{A}\right)=P(B)\times P(A)-P(A\cap B)\times P(A)\\
&\Leftrightarrow P(A\cap B)\times P\left(\overline{A}\right)+P(A\cap B)\times P(A)=P(B)\times P(A)\\
&\Leftrightarrow P(A\cap B)\times\Big(P\left(\overline{A}\right)+P(A)\Big)=P(B)\times P(A)\\
&\Leftrightarrow P(A\cap B)\times 1=P(B)\times P(A) && (4)\\
&\Leftrightarrow P(A\cap B)=P(B)\times P(A)
\end{aligned}$$

(1) Definição: $P(X|Y)=\dfrac{P(X\cap Y)}{P(Y)}$
(2) $P(A)>0$, e como $P(A)<1$, então $P\left(\overline{A}\right)>0$
(3) Teorema: $P\left(X\cup\overline{Y}\right)=P(X)-P(X\cap Y)$
(4) Teorema: $P\left(\overline{X}\right)+P(X)=1$

Logo, $P(A\cap B)=P(A)\times P(B)\ \Leftrightarrow\ P(B|A)=P\left(B|\overline{A}\right)$ *q.e.d.*

Exame – 2004, Ép. especial

21. Temos que,

$$\begin{aligned}
P\left(\overline{A}\right)-P(B)+P(A|B)\times P(B) &= P\left(\overline{A}\right)-P(B)+\frac{P(A\cap B)}{P(B)}\times P(B) && (1)\\
&= P\left(\overline{A}\right)-P(B)+P(A\cap B)\\
&= 1-P(A)-P(B)+P(A\cap B)\\
&= 1-\Big(P(A)+P(B)-P(A\cap B)\Big) && (2)\\
&= 1-P(A\cup B) && (3)\\
&= P\left(\overline{A\cup B}\right) && (2)\\
&= P\left(\overline{A}\cap\overline{B}\right) && (4)
\end{aligned}$$

(1) Definição: $P(X|Y)=\dfrac{P(X\cap Y)}{P(Y)}$
(2) Teorema: $P\left(\overline{X}\right)=1-P(X)$
(3) Teorema: $P(X\cup Y)=P(X)+P(Y)-P(X\cap Y)$
(4) Leis de De Morgan: $\overline{X}\cap\overline{Y}=\overline{X\cup Y}$

Logo, $P\left(\overline{A}\cap\overline{B}\right)=P\left(\overline{A}\right)-P(B)+P(A|B)\times P(B)$ *q.e.d.*

Exame – 2002, 1ª Fase – 1ª chamada

22. Temos que,

$$\begin{aligned}
1-P(E_1)\times P(E_2|E_1) &= 1-P(E_1)\times\frac{P(E_2\cap E_1)}{P(E_1)} && \text{Definição: } P(X|Y)=\frac{P(X\cap Y)}{P(Y)}\\
&= 1-P(E_2\cap E_1) &&\\
&= P\left(\overline{E_2\cap E_1}\right) && \text{Teorema: } P\left(\overline{X}\right)=1-P(X)\\
&= P\left(\overline{E_2}\cup\overline{E_1}\right) && \text{Leis de De Morgan: } \overline{X\cap Y}=\overline{X}\cup\overline{Y}
\end{aligned}$$

Logo, $P\left(\overline{E_1}\cup\overline{E_2}\right)=1-P(E_1)\times P(E_2|E_1)$ *q.e.d.*

Exame – 2000, 2ª Fase

23. Temos que,

$$\begin{aligned}
P(A)+P(B)+P\left(\overline{A}\cap\overline{B}\right) &= P(A)+P(B)+P\left(\overline{A\cup B}\right) && \text{Leis de De Morgan: } \overline{X}\cap\overline{Y}=\overline{X\cup Y}\\
&= P(A)+P(B)+1-P(A\cup B) && \text{Teorema: } P\left(\overline{X}\right)=1-P(X)\\
&= 1+P(A)+P(B)-P(A\cup B) &&\\
&= 1+P(A\cap B) && \text{Teorema: } P(X\cap Y)=P(X)+P(Y)-P(A\cup Y)
\end{aligned}$$

Logo, $P(A)+P(B)+P\left(\overline{A}\cap\overline{B}\right)=1+P(A\cap B)$ *q.e.d.*

Prova modelo – 2000